THE DECEMBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY AT CHICAGO.

THE thirty-eighth regular meeting of the Chicago Section, being the seventh regular meeting of the Society at Chicago, took place at the University of Chicago on Friday, December 22. The meeting was attended by fifty-eight persons, a number only slightly below the usual total of attendance registrations. In view of the fact that the date of this meeting was very inconvenient, coming during the last days preceding Christmas, this nearly normal attendance may be looked upon as very encouraging. The following forty-two members of the Society were present:

Dr. Florence E. Allen, Professor F. Anderegg, Professor G. N. Bauer, Professor A. A. Bennett, Professor G. A. Bliss, Professor Henry Blumberg, Professor H. T. Burgess, Professor W. D. Cairns, Professor R. D. Carmichael, Mr. E. H. Clarke, Professor H. E. Cobb, Professor G. H. Cresse, Professor D. R. Curtiss, Dr. W. W. Denton, Professor L. E. Dickson, Professor L. W. Dowling, Professor Arnold Dresden, Professor Arnold Emch, Professor W. B. Ford, Professor A. S. Hathaway, Professor T. H. Hildebrandt, Mr. Glenn James, Professor A. M. Kenyon, Dr. W. W. Küstermann, Professor Kurt Laves, Professor A. C. Lunn, Professor H. W. March, Professor William Marshall, Professor E. H. Moore, Professor C. C. Morris, Professor E. J. Moulton, Professor F. R. Moulton, Dr. J. R. Musselman, Miss I. M. Schottenfels, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor C. H. Sisam, Professor H. E. Slaught, Professor E. J. Townsend, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor J. W. A. Young.

The chairman of the Section, Professor W. B. Ford, presided at the two sessions. In the evening forty-seven members and friends dined together at the Quadrangle Club.

Plans are being made for including in the programme of the spring meeting a symposium on the subject of "The Lebesgue integral," to which an entire afternoon session is to be devoted.

The following papers were presented at this meeting:

(1) Dr. G. E. WAHLIN: "The cubic equation in a p-adic domain."

(2) Mr. GLENN JAMES: "Certain restrictions upon methods of summing divergent series."

(3) Dr. A. R. SCHWEITZER: "Some remarks concerning quasi-transitive functional equations."

(4) Professor H. T. BURGESS: "Note on the solution of the matrix equation $X^{-1}AX = N$."

(5) Professor HENRY BLUMBERG: "Certain general properties of functions."

(6) Professor A. S. HATHAWAY: "Certain residue groups."

(7) Professor E. B. VAN VLECK: "A proof of Haskins' momental theorem."

(8) Professor G. A. BLISS: "A device applicable to the problems of Lagrange and Mayer in the calculus of variations."

(9) Professor C. H. SISAM: "On the order of surfaces generated by systems of algebraic curves."

(10) Professor L. E. DICKSON: "On the history of the theory of numbers (preliminary communication)."

(11) Professor F. R. MOULTON: "An extended solution of differential equations."

(12) Professors G. N. BAUER and H. L. SLOBIN: "A system of algebraic and transcendental equations."

(13) Professor A. A. BENNETT: "Closed algebraic correspondences."

Dr. Wahlin's paper was read by title. Abstracts, numbered to correspond to the titles in the list above, follow below.

1. In this paper Dr. Wahlin makes a study of the general cubic equation for the domain of a rational prime p. He determines the conditions which the coefficients must satisfy in order that the cubic shall be irreducible or shall have one or three roots in $k(p)$. The result is then used and the necessary extension made for the complete determination of the factorization of p in the cubic domain.

2. Mr. James considers some of the necessary restrictions upon any method of summing series by the use of convergence factors, the assumption being made that the method satisfies "the conditions of consistency." In particular he proves that the limit of the sum of the convergence factors is unity, provided the method sums at least one convergent series, whose sum is not zero, to its ordinary sum and the limit of each of

the first K factors is zero. In the second part of his paper he shows that $\lim_{x=1} \Sigma a_n x^n$ cannot exist for a certain class of oscillating series.

3. Referring to a previous* paper, Dr. Schweitzer points out that it is possible to replace, in each theorem there given, all functional equations (in the hypothesis) but one by certain rational conditions on the M's. In particular, in the cubic case, the theorem as stated by the author (l. c.) remains valid if one of the functional equations is dropped from the hypothesis.

As subcategories of categories of functional equations previously† defined, the author defines (1) the quasi-transitive of degree $(n+1)$ $(n = 1, 2, 3, \cdots)$ and order k, $0 < k \leqq n+1$, (2) the transitive of degree $(n+1)$ and order k, $0 < k \leqq n+1$. The degree is defined as the number of arguments of any one of the functions involved in the equation; the order is the degree less the number of variables eliminated. If the order equals the degree, the concepts "quasi-transitive" and "transitive" coalesce and one has as class subordinate to the resulting class the "iterative and non-eliminative" equation. Examples: $\phi\{f(t, x_2, x_3), f(x_1, t, x_3), f(x_1, x_2, t)\} = f(x_1, x_2, x_3)$ is a transitive equation of degree three and order two; $\phi\{f(t_1, t_2, x_3), f(t_1, x_2, t_2), f(x_1, t_1, t_2)\} = f(x_1, x_2, x_3)$ is a transitive equation of degree three and order one; $f\{f(x_1, x_2, x_3), f(x_2, x_3, x_1), f(x_3, x_1, x_2)\} = f(x_1, x_2, x_3)$ is an iterative equation of degree three and order three.

The preceding classification of equations on the basis of order and degree necessarily involves the finiteness of the number of variables. It appears possible, however, to define at least formally extensions of categories of functional equations previously defined by the author to equations of infinite degree. In particular, one can give formal definitions of transitive and quasi-transitive equations of infinite degree and order k where k is a positive integer. For example, $\phi\{u_1, u_2, \cdots, u_n, \cdots\} = \psi(x_1, x_2, \cdots, x_n, \cdots)$ where $u_i = \lambda_i f(x_i, t_1, t_2, \cdots, t_n, \cdots)$ $(i = 1, 2, \cdots)$ is of infinite degree and order 1. The domain of validity of any of these formally defined equations is, of course, a matter of special investigation.

* Cf. BULLETIN, vol. 23 (1916), pp. 76–78.

† Cf. BULLETIN, previous abstracts.

On the basis of the properties of an abstract field, one defines directly an abstract relation having $n+1$ elements as arguments ($n = 1, 2, \cdots$) and whose functional interpretation is the equation $f\{u_1, u_2, \cdots, u_{n+1}\} = f(x_1, x_2, \cdots, x_{n+1})$, where $u_i = f\{t_1, t_2, \cdots, t_n, x_i\}$ $(i = 1, 2, \cdots, n+1)$. In the latter relation $f(x_1, x_2, \cdots, x_{n+1}) = f_1\{u_{123\cdots n}, u_{2[n+1]}\}$, $n \geqq 2$, $u_{12} = f_1(x_1, x_2)$, $u_{123} = f_1(u_{12}, u_{23})$, $\cdots$, $u_{123\cdots n} = f_1(u_{123\cdots[n-1]}, u_{2n})$, and $f_1\{f_1(t, x_2), f_1(t, x_1)\} = f_1(x_1, x_2)$.

4. If A is any square matrix, there exists a similar matrix N which is of a well-known canonical form. In a paper in the *Annals of Mathematics*, September, 1916, Professor Burgess gave a practical method of determining the form of N for any given matrix A. In the present paper he gives a practical method for computing the matrix X such that $X^{-1}AX = N$.

The existence of X is well known, but no direct method computing it is to be found in the literature.

5. A résumé of Professor Blumberg's paper is contained in the *Proceedings of the National Academy of Sciences*, November, 1916. The paper will appear in full in the *Annals of Mathematics*.

6. Mr. Hathaway groups the residues of a prime $n = 6m + 2\epsilon + 3$ $(\epsilon = 1$ or $2)$, so that $x = a$, $1/a$, $-1-a$, $-1/(1+a)$, $-(1+a)/a$, $-a/(1+a)$, make a group. There are m groups of six residues, one of three residues $(1, -2, -\frac{1}{2})$, and ϵ of two residues, $(0, -1)$ and, when $\epsilon = 2$, $(\omega, \omega^2 = -1-\omega)$.

The function $y = (1+x+x^2)^3/(x+x^2)^2$ is invariant for a group, and designating the values for the $m+1$ groups by y_r, $r = 0, 1, \cdots, m$, in which $y_m = 27/4$, the invariant of the group of three, the numbers of quadratic and non-quadratic residues y_r are such that we may assign odd subscripts to the first, and even subscripts to the second.

Taking $\alpha = 3(m-r)+\epsilon$, $\beta = 2r+1$, $A_r = (2\alpha 3\beta)/n = \Gamma(\alpha+\beta)/\Gamma(\alpha+1)\Gamma(\beta+1)$, to define the functions, $\varphi(y) = \Sigma A_r y^{m-r}$, $F(y) = \Sigma\beta A_r y^{m-r}$, $F_1(y) = \Sigma\beta A_r y^{m-r}/(\beta+1)$, the congruences $F(y) \equiv 0$, $F_1(y) \equiv 0$, $8y^{m+1} - 3F_1(y) \equiv 0$, mod. n, are completely resolvable in y_r residues, the first in all except y_m, the second and third in quadratic and non-quadratic y's respectively.

If the sum of three nth powers be divisible by n^2, one of the powers, or the difference of the cubes of two of them, or the resultant of $\varphi(y) = 0$, $F(y) = 0$ is divisible by n.

7. Professor Van Vleck's paper gives a proof of the following theorem of Haskins (*Transactions*, April, 1916): Let $f(x)$ and $\phi(x)$ be two functions which are integrable in the sense of Lebesgue over the interval (a, b) and have upper and lower bounds,

$$h \leqq f(x) \leqq H, \qquad h \leqq \phi(x) \leqq H;$$

then if

$$\int_a^b [f(x)]^n dx = \int_a^b [\phi(x)]^n dx \qquad (n = 1, 2, 3, \cdots),$$

the measures of the sets of points $E_f(\alpha, \beta)$ and $E_\phi(\alpha, \beta)$ for which $f(x)$ and $\phi(x)$ lie between α and β inclusive will be equal for all pairs of values (α, β) included in the function range (h, H). Professor Van Vleck reduces this theorem to a special case of the problem of moments of Stieltjes and then establishes it by this author's results.

8. A class of curves in space of $n + 1$ dimensions may be defined by the conditions that each curve of the class must satisfy a set of differential equations of the form

$$\varphi_\alpha(x; y_1, \cdots, y_n; y_1', \cdots, y_n') = 0 \quad (\alpha = 1, 2, \cdots, m < n)$$

and pass through two fixed points. The problem of Lagrange is to characterize a curve of the class which makes an integral

$$\int f(x; y_1, \cdots, y_n; y_1', \cdots, y_n')dx$$

a maximum or minimum. It is customary to assume that along the particular curve in question the determinant of the first m rows of the matrix $||\partial\varphi_\alpha/\partial y_k'||$ $(\alpha = 1, \cdots, m; k = 1, \cdots, n)$ is everywhere different from zero. This is an unfortunately unsymmetric assumption, and it is the purpose of the paper of Professor Bliss to show that the details of the theory can be carried through if the matrix is only assumed to be of rank m at each point of the minimizing or maximizing curve. The device used is the addition of $n - m$ variables z_r $(r = m + 1, \cdots, n)$, and $n - m$ differential equations

$$\varphi_r(x; y_1, \cdots, y_n; y_1', \cdots, y_n') = z_r \quad (r = m + 1, \cdots, n),$$

such that the determinant $|\partial\varphi_i/\partial y_k'|$ $(i, k = 1, \cdots, n)$ is everywhere different from zero on the curve to be studied. The results are of course independent of the auxiliary equations and variables introduced. A similar method is applicable to the problem of Mayer.

9. Professor Sisam's paper discusses the relation between the order of the curves of a system of algebraic curves on an algebraic surface, the number of variable points common to two generic curves of the system, and the order of the surface. It determines a new sufficient condition that such a system of curves constitute a pencil.

11. In general, the proofs of the existence of the solutions of differential equations establish their existence only in limited domains. In this paper Professor Moulton gives a practical method, related closely to that known as Picard's method, for constructing the solutions so long as the independent and dependent variables remain in the region for which the differential equations have certain specified properties. By suitable specialization, the process reduces to the Cauchy-Lipschitz method. The validity of the method of mechanical quadratures employed by astronomers and physicists is established. Finally, corresponding results are established for a certain type of an infinite system of differential equations in infinitely many variables.

12. In the present paper Professors Bauer and Slobin study a system of three equations of the form

$$(A) \qquad \varphi_1(x, y) = 0, \quad \varphi_2(x, y) = 0, \quad T(x, y) = 0,$$

where the φ_i's are polynomials in x and y with algebraic coefficients, and where $T(x, y) = 0$ denotes an equation which, in general, is not satisfied if x and y are both algebraic numbers. As illustrations of equations of this type, may be mentioned

$$x - e^y = 0, \quad y - \sin x = 0, \quad \varphi_1(x, y) + \varphi_2(x, y) \tan \varphi_3(x, y) = 0.$$

$$\varphi_1(x, y)e^{\varphi_2(x, y)} + \varphi_3(x, y) = 0.$$

The following are some of the theorems considered:

1. The system A cannot have simultaneous solutions (excepting possibly for the pairs of algebraic numbers, if any,

which satisfy $T(x, y) = 0$). No curves represented by equations of the form $\varphi_1(x, y) = 0$, $\varphi_2(x, y) = 0$ can intersect on the sine curve or the tangent curve excepting at the origin, nor on the exponential curve $x = e^y$ excepting in the point (1, 0).

2. The system $\varphi_1(x, y) + p(t)\varphi_2(x, y) = 0$, $\varphi_3(x, y) = 0$, $\varphi_4(x, y) = 0$ can have no simultaneous solutions except those, if any, which simultaneously satisfy $\varphi_1 = 0$, $\varphi_2 = 0$, $\varphi_3 = 0$, $\varphi_4 = 0$, where $p(t)$ is a polynomial in t, a transcendental number.

3. All the singularities of the curves represented by $\psi(x, y) \equiv \varphi_1(x, y) + t\varphi_2(x, y) = 0$ which require $\partial\psi/\partial x = 0$ and $\partial\psi/\partial y = 0$, lie upon $\varphi_1(x, y) = 0$ and $\varphi_2(x, y) = 0$ if there are any at all.

4. All singularities of $\varphi_0(x, y) + p_1(t)\varphi_1(x, y) + \cdots + p_n(t)\varphi_n(x, y) = 0$ must lie upon each of the curves represented by the equations $\varphi = 0$, $\varphi_1 = 0$, $\cdots$, $\varphi_n = 0$ where $p_i(t)$, $(i = 1, \cdots, n)$, are polynomials in t, a transcendental number, the coefficients of the polynomials being algebraic numbers.

13. In this paper, Professor Bennett examines by elementary methods the form of a closed algebraic correspondence upon an algebraic curve or Riemann surface. As a result of certain elementary properties of involutions, and the group properties of closed algebraic and finite systems of points, the relations between closed correspondences and variable inscribed plane configurations are described. A systematic method of securing fundamentally different generalizations of the closure problems arising in connection with Poncelet polygons is obtained.

ARNOLD DRESDEN,
Secretary of the Section.

THE TWENTY-THIRD ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-third annual meeting of the Society, which was held in New York City on Wednesday and Thursday, December 27–28, 1916, was in several respects an exceptional occasion. It took place in the midst of the convocation week series of meetings of the American association for the advance-

ment of science and its long train of affiliated societies; and was immediately followed by the second annual meeting of the newly organized Mathematical association of America, with which the Society has not only a large common membership but also a general community of interest highly beneficial to both. The annual meeting is always one of the largest of the year, being the season of the election of officers and other members of the Council and the transaction of important business. This year it was especially marked by the delivery of the retiring address of President E. W. Brown, the subject of which was "The relations of mathematics to the natural sciences." This was presented before a joint session of the Society with the Mathematical association, the Astronomical society of America, and Section A of the American association and was followed by the retiring address of Vice-President A. O. Leuschner of Section A, on "Derivation of orbits—theory and practice." A joint dinner of the four organizations was held on Thursday evening at the Park Avenue Hotel, with an attendance of 143 members and friends. Much of the credit for the great success of the meetings is due to the joint committee on arrangements and to the programme committees of the Mathematical association.

Under all these favorable circumstances the attendance at the four sessions of the Society exceeded all previous records. It included the following 131 members and perhaps others whom the Secretary failed to identify in the multitude:

Professor R. B. Allen, Professor R. C. Archibald, Professor C. H. Ashton, Professor Clara L. Bacon, Dr. Charlotte C. Barnum, Professor R. D. Beetle, Professor G. D. Birkhoff, Professor Joseph Bowden, Professor J. W. Bradshaw, Professor E. W. Brown, Dr. T. H. Brown, Professor Daniel Buchanan, Professor W. G. Bullard, Dr. R. W. Burgess, Professor W. D. Cairns, Professor Florian Cajori, Professor W. B. Carver, Professor A. B. Coble, Professor Abraham Cohen, Professor F. N. Cole, Professor J. L. Coolidge, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Professor C. H. Currier, Professor T. W. Edmondson, Professor L. P. Eisenhart, Professor T. C. Esty, Professor G. C. Evans, Mr. G. W. Evans, Professor F. C. Ferry, Professor J. C. Fields, Professor H. B. Fine, Dr. C. A. Fischer, Professor T. S. Fiske, Professor T. M. Focke, Professor W. B. Ford, Professor A. S. Gale, Professor W. A. Garrison, Professor O. E. Glenn, Mr. T. E. Gravatt,

Dr. G. M. Green, Dr. T. H. Gronwall, Professor C. C. Grove, Professor H. V. Gummere, Dr. W. L. Hart, Dr. Mary G. Haseman, Professor M. W. Haskell, Professor H. E. Hawkes, Dr. Olive C. Hazlett, Professor E. R. Hedrick, Dr. A. A. Himwich, President C. S. Howe, Professor L. A. Howland, Professor L. S. Hulburt, Professor E. V. Huntington, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Dr. E. A. T. Kircher, Dr. J. R. Kline, Mr. E. H. Koch, Jr., Professor J. K. Lamond, Mr. Harry Langman, Professor Florence P. Lewis, Professor G. H. Ling, Dr. Joseph Lipka, Professor W. R. Longley, Dr. J. V. McKelvey, Professor Emilie N. Martin, Professor Helen A. Merrill, Dr. Mansfield Merriman, Professor E. J. Miles, Dr. A. L. Miller, Professor Bessie I. Miller, Professor G. A. Miller, Professor H. B. Mitchell, Professor H. H. Mitchell, Professor R. L. Moore, Professor F. M. Morgan, Professor Frank Morley, Professor Richard Morris, Mr. G. W. Mullins, Professor G. D. Olds, Professor F. W. Owens, Mr. George Paaswell, Dr. Alexander Pell, Professor Anna J. Pell, Dr. G. A. Pfeiffer, Professor A. D. Pitcher, Professor J. M. Poor, Professor H. W. Reddick, Professor R. G. D. Richardson, Mr. J. F. Ritt, Professor E. D. Roe, Jr., Professor R. E. Root, Dr. J. T. Rorer, Professor D. A. Rothrock, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Mary E. Sinclair, Professor H. E. Slaught, Professor Clara E. Smith, Professor D. E. Smith, Professor P. F. Smith, Professor Sarah E. Smith, Professor W. M. Smith, Professor Virgil Snyder, Professor Pauline Sperry, Dr. J. M. Stetson, Professor Henry Taber, Professor H. D. Thompson, Dr. J. I. Tracey, Professor C. B. Upton, Professor J. N. Van der Vries, Mr. H. S. Vandiver, Mr. C. E. Van Orstrand, Professor Oswald Veblen, Dr. J. H. Weaver, Mr. H. E. Webb, Professor A. G. Webster, Dr. Mary E. Wells, Professor H. S. White, Professor E. E. Whitford, Mr. J. K. Whittemore, Professor A. H. Wilson, Professor E. B. Wilson, Professor Ruth G. Wood, President R. S. Woodward, Professor J. W. Young, Dr. Mabel M. Young.

President Brown occupied the chair, being relieved by Vice-Presidents Hedrick and Snyder and by Professor Olds. The Council announced the election of the following persons to membership in the Society: Professor H. H. Conwell, University of Idaho; Mr. Robert Dysart, Boston, Mass.; Dr. Mary G. Haseman, Johns Hopkins University; Mr. J. B.

Scarborough, North Carolina Agricultural and Mechanical College; Mr. J. J. Tanzola, U. S. Naval Academy. Ten applications for membership in the Society were received.

In response to an invitation received from the Department of Mathematics of the University of Chicago, it was decided to hold the summer meeting and colloquium of the Society at that university in 1919. A committee was appointed to arrange for the summer meeting of 1917. A committee was also appointed to proceed with the early publication of the Cambridge Colloquium Lectures; the volume will probably appear in the early summer.

The total membership of the Society is now 732, including 75 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 490; the number of papers read was 205. The number of members attending at least one meeting during the year was 278. At the annual meeting 235 votes were cast. The Treasurer's report shows a balance of $10,198.38, including the life membership fund of $6,039.87. Sales of the Society's publications during the year amounted to $1,434.28. The Library now contains 5,377 volumes, excluding unbound dissertations.

At the annual election, which closed on Thursday morning, the following officers and other members of the Council were chosen:

President,	Professor L. E. DICKSON.
Vice-Presidents,	Professor A. B. COBLE, Professor E. B. WILSON.
Secretary,	Professor F. N. COLE.
Treasurer,	Professor J. H. TANNER.
Librarian,	Professor D. E. SMITH.

Committee of Publication,

Professor F. N. COLE,
Professor VIRGIL SNYDER,
Professor J. W. YOUNG.

Members of the Council to Serve until December, 1919,

Professor G. C. EVANS,
Professor G. H. LING,
Professor L. A. HOWLAND,
Professor R. L. MOORE.

The following papers were read at the annual meeting:

(1) Professor J. E. ROWE: "The relation of singularities of the rational quintic in space to loci of the rational plane quintic."

(2) Dr. C. A. FISCHER: "Linear functionals of n-spreads."

(3) Professor H. B. MITCHELL: "Geometrical limits for the imaginary roots of a polynomial with real coefficients."

(4) Professor ARNOLD EMCH: "A theorem on the curves described by a spherical pendulum."

(5) Mr. J. K. WHITTEMORE: "Spiral minimal surfaces."

(6) Dr. J. R. KLINE: "Concerning the complement of a countable infinities of point sets of a certain type."

(7) Professor L. L. DINES: "On projective transformations in function space."

(8) Professor C. C. GROVE: "Foundation of the correlation coefficient."

(9) Professor O. E. GLENN: "Preliminary report on invariant systems belonging to domains."

(10) Dr. NORBERT WIENER: "Certain formal invariances in Boolean algebras."

(11) Professor L. P. EISENHART: "Theory of transformations T of conjugate systems."

(12) Professor E. V. HUNTINGTON: "Complete existential theory of the postulates for serial order."

(13) Professor E. V. HUNTINGTON: "Complete existential theory of the postulates for well ordered sets."

(14) Professor DANIEL BUCHANAN: "Orbits asymptotic to an isosceles-triangle solution of the problem of three bodies."

(15) Professor DANIEL BUCHANAN: "Asymptotic satellites about the straight-line equilibrium points."

(16) Professor DANIEL BUCHANAN: "Asymptotic satellites about the equilateral-triangle equilibrium points."

(17) Dr. G. M. GREEN: "Isothermal nets on a curved surface."

(18) Dr. A. L. MILLER: "Systems of pencils of lines in ordinary space."

(19) Dr. W. L. HART: "On an infinite system of ordinary differential equations."

(20) Dr. W. L. HART: "Linear differential equations in infinitely many variables."

(21) Professor E. V. HUNTINGTON: "A set of independent postulates for cyclic order."

(22) Professor E. V. HUNTINGTON: "Sets of independent postulates for order on a closed line."

(23) Professor FRANK MORLEY: "The cubic seven-point and the Lüroth quartic."

(24) Professor J. L. COOLIDGE: "The intersections of a straight line and hyperquadric."

(25) Professor A. D. PITCHER: "Biextremal connected sets."

(26) Professor H. H. MITCHELL: "Proof that certain ideals in a cyclotomic realm are principal ideals."

(27) Professor H. H. MITCHELL: "On the asymptotic value of sums of power residues."

(28) Professor EDWARD KASNER: "Certain systems of curves connected with the theory of heat."

(29) Miss TERESA COHEN: "On a concomitant curve of the planar quartic."

(30) Professor P. F. SMITH: "A theorem for space analogous to Cesàro's theorem for plane isogonal systems."

(31) Professor W. E. STORY: "Some variable three-term scales of relation."

(32) Professor E. W. BROWN: presidential address: "The relations of mathematics to the natural sciences."

(33) Professor A. O. LEUSCHNER, vice-presidential address, Section A, American association for the advancement of science: "Derivation of orbits—theory and practice."

(34) Professor G. D. BIRKHOFF: "A class of series allied to Fourier series."

(35) Professor G. D. BIRKHOFF: "Note on linear difference equations."

(36) Professor G. A. MILLER: "Groups generated by two operators of the same prime order such that the conjugates of the one under the powers of the other are commutative."

(37) Mr. H. S. VANDIVER: "On the power characters of units in a cyclotomic field."

(38) Professor HENRY TABER: "On the structure of finite continuous groups."

Dr. Wiener was introduced by Professor Huntington, Miss Cohen by Professor Morley. Professor Leuschner's vice-presidential address was read by Professor Haskell. The papers of Professor Rowe, Professor Emch, Professor Dines, the first two papers of Professor Buchanan, Dr. Hart's first paper, and the papers of Professor Story and Professor Taber were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In a previous paper (*Transactions*, volume 13, No. 3, pages 395–404) Professor Rowe has pointed out how cocombinant curves of the rational plane quintic R_2^5, whose parametric equations are written *with* binomial coefficients, may be transformed into the parameters of singularities on the rational space quintic R_3^5, whose parametric equations are written *without* binomial coefficients. For instance, it was shown there that the point projection of a point of the plane upon the R_2^5 transforms into the parameters of the eight tetratactic planes of the R_3^5.

The present paper is a completion of part of the work suggested in the paper cited above. The simplest singularities of a rational plane curve are flex tangents, double tangents, and double points. The simplest singularities of a rational space curve are tetratactic planes, touching-osculating planes, tritangent planes, unisecant-tangent lines (i. e., lines tangent to the curve at one point and meeting it in one other point), and quadrisecant lines. Obviously the R_3^5 does not have any tritangent planes, but it has all of the other singularities. The author has shown how to derive the parameters of these from the proper covariant loci of the R_2^5.

The new results obtained may be stated briefly as follows: (1) The quartic (20), page 399 of the paper referred to above, transforms into a quartic whose roots are the parameters of the points cut out of the R_3^5 by its unique quadrisecant line. (2) Through every point of the plane pass twelve osculant conics of the R_2^5; the twelvic yielding the parameters of these transforms into the twelvic whose roots are the parameters of the twelve touching-osculating planes of the R_3^5. (3) Through every point of the plane pass twelve flex tangents of twelve osculant cubics of the R_2^5; the twelvic whose roots are the parameters of these cubics transforms into the twelvic whose roots are the parameters of the twelve unisecant-tangent lines of the R_3^5.

2. It has been proved by Riesz that a linear functional of a curve which is continuous with zeroth order can be expressed as a Stieltjes integral. The multiple Stieltjes integral has been defined by Fréchet, and proved to be a linear functional.

In the present paper Dr. Fischer has found some additional properties of such integrals, and proved that any linear functional of an n-spread which is continuous with zeroth order can be expressed as a multiple Stieltjes integral.

3. At the October meeting of the Society, Professor Mitchell presented a number of criteria whereby geometrical limits for the position of a pair of imaginary roots of an algebraic polynomial with real coefficients might be determined by inspection from the relative position of its real roots and bend points. In the present paper the method is extended so as to yield precise geometrical constructions when the polynomial has but two imaginary roots, and, in the case of more than two, to isolate the separate roots geometrically within regions that can be made small at will.

4. Professor Emch's paper appeared in full in the February BULLETIN.

5. It is well known that corresponding points of a family of associated minimal surfaces lie on an ellipse. In another paper Mr. Whittemore has found the locus of the vertices of these ellipses for any such family, and shown that, if the minimal surfaces are applicable to a surface of revolution, the locus consists of two parts, a plane and a surface of revolution. In the present paper it is proved that the only other minimal surfaces for which part of the locus is a plane are the spiral minimal surfaces, and that in this case the rest of the locus is a certain spiral surface. The author has then made a study of spiral minimal surfaces and proved certain properties of these surfaces, in many cases analogous to properties of minimal surfaces applicable to surfaces of revolution. These properties are related to the spiral curves of the surfaces, and concern lines of curvature, asymptotic lines, double lines, evolute surfaces, etc.

6. Dr. Kline's paper appears in full in the present number of the BULLETIN.

7. The transformations considered by Professor Dines are of the form

$$(1) \qquad f_1(x) = \frac{a(x) + b(x)f(x) + \int_0^1 c(x, y)f(y)dy}{d + \int_0^1 e(y)f(y)dy},$$

where the functions a, b, c, d, e satisfy the conditions:

1. All are continuous for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$;
2. The absolute value of $b(x)$ is greater than a positive constant on the interval $0 \leqq x \leqq 1$;
3. The equation $a(x) + b(x)f(x) + \int_0^1 c(x, y)f(y)dy = 0$ has a unique continuous solution $f(x) = h(x)$;
4. $d + \int_0^1 e(y)h(y) \neq 0$.

The set of all transformations of this type form a group under which the lines of function space are interchanged among themselves. Properties analogous to the properties of the corresponding projective transformations in space of n dimensions are obtained. Among other things it is shown that every finite transformation generated by the general infinitesimal projective transformation in function space, defined by Kowalewski, can be expressed in the form (1).

8. An analysis of the work of educational psychologists along the line of transfer and of the mathematics they have employed has led to the careful study of the foundations of the correlation coefficient, and cognate statistical measures discussed in this paper by Professor Grove.

9. Professor Glenn shows that, for a binary form, under general binary transformations, there exist complete systems of concomitants belonging to each of several domains of rationality. The methods by which the systems for these domains are constructed in case of the general transformation apply also to concomitants of substitutions constituting a special subset of the general transformations.

10. Dr. Wiener proves that the only operations in a Boolean algebra which possess formal properties such as logical addition and multiplication must possess are derivable from these by a one-one transformation belonging to the algebra. It is shown that negation remains invariant under all such transformations, and hence occupies a unique formal position in the algebra. The conditions under which an operation possesses such a unique formal position are discussed. The paper appeared in full in the January number of the *Transactions*.

11. When there is a one-to-one correspondence between the points of two surfaces S and S_1 of such a sort that the developables of the congruence of lines joining corresponding points meet S and S_1 in a conjugate system of curves on these surfaces, then S and S_1 are said to be in the relation of a transformation T. In several former papers Professor Eisenhart discussed certain types of these transformations, in particular those for which the conjugate systems on the two surfaces had their point invariants or their tangential invariants equal. In the present paper he develops the general theory of transformations T of a surface defined in either point or tangential coordinates. The well-known transformations D_m of isothermic surfaces are transformations T. The general transformations T admit a theorem of permutability reducing for the case of transformations D_m to the theorem due to Bianchi for the latter. The paper appeared in full in the January number of the *Transactions*.

12–13. Professor Huntington's two papers appear in full in the present number of the BULLETIN.

14. If two finite spheres of equal masses revolve about their common center of gravity, then an infinitesimal body subject to their attraction will describe a periodic orbit, under suitable initial conditions. This orbit is linear, being in the straight line through the center of gravity of the finite bodies and perpendicular to the plane of their motion. In a former paper (*Proceedings of the London Mathematical Society*, July, 1915), Professor Buchanan discussed the periodic oscillations about this linear orbit when the infinitesimal body is given an initial displacement from it. In the present paper, orbits are determined which are asymptotic to this linear orbit. The solutions are expansible as power series in a parameter which may denote the initial distance of the infinitesimal body from the linear orbit.

15. In this paper Professor Buchanan discusses orbits which are asymptotic to the periodic orbits of Class A and of Class B of the oscillating satellites about the straight-line equilibrium points as determined by Moulton in his Periodic Orbits, Chapter V.

16. In his third paper, Professor Buchanan discusses the

asymptotic satellites about the Lagrangian equilateral-triangle equilibrium points of the problem of three bodies. In Chapter IX of Moulton's Periodic Orbits, Buck determined two and three-dimensional periodic orbits about each equilibrium point. The two-dimensional periodic orbits were shown to exist only when one of the finite bodies is relatively small with respect to the other, but this restriction is not necessary for the existence of the three-dimensional orbits. In the present paper, however, it is shown that both the two and three-dimensional asymptotic orbits exist only when the finite bodies are more nearly equal than in the case of the two-dimensional periodic orbits. The two-dimensional asymptotic orbits approach the equilibrium points and not the corresponding periodic orbits. The three-dimensional asymptotic orbits, however, approach the three-dimensional periodic orbits.

17. An isothermal net on a curved surface is usually described as a net which divides the surface into infinitesimal squares. This of course is not properly speaking a geometric characterization of isothermal nets. Dr. Green gives two purely geometric characterizations of isothermal nets, one of which is easy to prove but requires a lengthy description; the other, however, though not so easy to prove, may be briefly described as follows. In previous papers Dr. Green has defined and made use of a certain geometric relation R, by means of which to a line l passing through any point of a curved surface is made to correspond a unique line l' in the corresponding tangent plane, the line l' being determined entirely by the line l and the parametric net of curves on the surface. Suppose the parametric net is any orthogonal net, and the line l is a normal to the surface. Then a unique line l' is determined in the corresponding tangent plane, so that corresponding to the congruence of normals l of the surface one obtains a congruence of lines l'. The developables of the congruence of lines l' correspond to a conjugate net on the surface, if and only if the parametric net is isothermal.

This obviously affords also an elegant characterization of isothermic surfaces, i. e., surfaces whose lines of curvature form an isothermal net. The other characterization of isothermal nets breaks down in this case; it also involves a use of the relation R, and the proof is so direct that the theorem

is interesting in spite of its failure when the lines of curvature are parametric.

18. The paper of Dr. Miller, in its three parts, treats of the projective differential properties of systems of ∞^1, ∞^2, and ∞^3 pencils of lines in ordinary space by means of the synthetic methods of Segre.

In part II the congruence determined by two nearby pencils of the system is found to be more interesting than, although closely allied to, the tangent linear congruence. The locus of the congruences determined by a pencil and all the infinitely near pencils of the system is found to be a tetrahedral complex of the type [(22) (11)]. Two interesting "developable" cases of these systems are discussed, one in which all the tangent linear congruences to the system at lines of a pencil have in common a pair of pencils with a common line, and one in which all these tangent congruences coincide. The locus of the centers of the pencils of the system and the envelope of their planes are found to be the focal surfaces of a director congruence to which all the pencils of the system are tangent in the first developable case, while in the second they coincide.

In part III the systems of ∞^3 pencils are classified according to the various kinds of two-parameter families of which they are the locus. This leads to a brief projective differential study of Pfaffian differential equations.

19. In a previous paper* Dr. Hart established the unique existence of a solution of the infinite system of ordinary differential equations

$$(1)\qquad \frac{dx_i}{dt} = f_i(x_1, x_2, \cdots; t) \qquad [x_i(t_0) = a_i; i = 1, 2, \cdots],$$

in which the variables were real and satisfied

$$(2)\qquad |t - t_0| \leqq b, \qquad |x_k - a_k| \leqq r_k < r.$$

In the former paper the results were obtained, by means of a generalization of the Picard approximation process, under the assumption that the f_i satisfied a continuity condition and, also, a condition with respect to the variables $(x_1, x_2, \cdots)$

* *Proc. National Academy of Sciences*, vol. 2 (1916), p. 309.

analogous to the classical Lipschitz condition. In the present paper there is established

Theorem I. Let the functions f_i be completely continuous in the region (2). Suppose that the maxima M_i (which exist) of the $|f_i(x_1, x_2, \cdots; t)|$ satisfy $M_i \leqq r_i M$. Then there exists *at least* one solution $\xi(t) = [x_1(t), x_2(t), \cdots]$ of (1), $x_i(t_0) = a_i$, which is defined and continuous for $t_0 \leqq t \leqq c$ where c is the smaller of b and $1/M$.

The solution is obtained by a generalization of the Cauchy polygon method; on adding the Lipschitz condition of the previous paper it is readily established that two solutions $\xi(t)$ cannot exist.

In the present paper Dr. Hart also treats a system of differential equations in general analysis. Among the particular instances of this general theory are found the case of a system of n differential equations

$$\frac{dx_i}{dt} = f_i(x_1, x_2, \cdots, x_n; t) \quad (i = 1, 2, \cdots, n),$$

where the f_i need not satisfy a Lipschitz condition, and also the case of system (1) under the hypotheses of Theorem I.

20. In considering finite systems of ordinary differential equations it is found that linear systems, in particular, have a great number of interesting properties. Many of these are connected with the notion of fundamental sets of solutions. Dr. Hart, in his second paper, considers a class of infinite systems of linear differential equations of the form

$$\frac{dx_i}{dt} = \sum_{j=1}^{\infty} k_{ij}(t)x_j + g_i(t) \quad (i = 1, 2, \cdots;]|t| \leqq r), \tag{1}$$

in which $(x_1, x_2, \cdots)$ are complex values for which $\sum_{i=1}^{\infty} |x_i|^2$ converges. The infinite matrix $(k_{ij}(t))_{i,\, j=1,\, 2,\, \ldots}$ is assumed to be *limited* in the sense used for the term by Hilbert. The $(g_1(t), g_2(t), \cdots)$ are such that $\sum_{i=1}^{\infty} |g_i(t)|^2$ converges for every t. The elements k_{ij} and g_i are assumed to be analytic in t and are regular for $|t| \leqq r$. For the system (1) an infinite matrix of solutions $A(t)$ is said to constitute a fundamental set of solutions in case A is limited and possesses a unique limited

reciprocal. The system (1), in relation to this notion of fundamental sets of solutions, is found to have many of the familiar properties of finite systems of linear equations.

21. Professor Huntington shows that a class K is arranged in cyclic order (that is, in circular order with a definite sense around the circle) by a triadic relation $R(ABC)$, if the following five postulates are satisfied: (A) If ABC, then BCA. (B) If A, B, C are distinct, then at least one of the six triads, ABC, ACB, etc., is true. (C) If ABC is true, then ACB is false. (D) If ABC is true, then A, B, C are distinct. (1) If A, B, X, Y are distinct, and XAB and AYB, then XAY. The independence of these five postulates is established, and the relations between these postulates and the postulates for betweenness (BULLETIN, volume 23, pages 70–71, November, 1916) is discussed. A fuller abstract of the present paper may be found in the *Proceedings* of the National Academy of Sciences, volume 2, pages 630–631, November, 1916.

22. The theory of order on a closed line without distinction of sense is the same as the theory of the separation of pairs of points (begun by Vailati in 1895), and is based on a tetradic relation $ABCD$. Professor Huntington starts with the following basic list of fourteen postulates: (A_1) If $ABCD$, then $BCDA$. (A_2) If $ABCD$, then $DCBA$. (B) If A, B, C, D are distinct, then at least one of the twenty-four possible permutations $ABCD$, $ABDC$, etc., is a true tetrad. (C) If $ABCD$ is true, then $ABDC$ is false. (D) If $ABCD$ is true, then A, B, C, D are distinct. If A, B, C, X, Y are distinct, and $ABXC$ and $ABCY$, then: (1) $ABXY$; (2) $BXCY$; (3) $AXCY$. If A, B, C, X, Y are distinct, and $ABCX$ and $ABCY$, then: (4) $ABXY$ or $ABYX$; (5) $ACXY$ or $ACYX$; (6) $BCXY$ or $BCYX$; (7) $ABXY$ or $ACYX$; (8) $ABXY$ or $BCYX$; (9) $ACXY$ or $BCYX$. From this basic list, which contains many redundancies, eight different sets of independent postulates are obtained as follows (postulates A_1, A_2, B, C, D being required in each set): (I) 1, 4; (II) 1, 5; (III) 1, 6; (IV) 1, 7; (V) 1, 8; (VI) 1, 9; (VII) 2; (VIII) 3. Of these eight sets, the seventh appears to be the only one previously known.

23. Professor Morley's paper is in abstract as follows: Aronhold's construction of a curve of class four from seven

given points a_i gives a quartic of Lüroth's type when the seven points have each the same polar line as to a conic and a cubic (Bateman, *American Journal*, volume 36, 1914). If so, the nodal tangents of a cubic with a double point a_i and on the other six points will be apolar to the conic. Therefore a skew symmetric form $a^5x^2y^2$, which when y is a_i is the nodal tangents, will be such that the 6-rowed determinant of its coefficients vanishes. This determinant is the square of a Pfaffian of degree 15 in a_i, and further the Pfaffian vanishes when six of the points are on a conic. There is then a *cubic* invariant of 7 points which vanishes when they are an Aronhold set of a Lüroth quartic. This is the only cubic invariant of seven points. For six given points, there is then a cubic curve, on the six. Mapping on a cubic surface we have 36 plane sections (one for each double six), the locus of points on the surface from which the tangent cones are of Lüroth's type. Hence the degree of Lüroth's invariant is 51.

24. Professor Coolidge's paper contains a symmetrical form for the solution of $n - 1$ linear and one quadratic equation in $n + 1$ homogeneous variables.

25. In his thesis Fréchet considers the theory of functions μ of a variable q which ranges over a general class Q on which is defined a distance function δ conditioned to meet the needs of the theory. Professor Pitcher considers such systems $(Q; \delta)$ where δ is such that $\delta(qq) = 0$ and $\delta(q_1q_2) = \delta(q_2q_1)$. In particular he gives a set of three very simple conditions on the system $(Q; \delta)$ which are completely independent and which are both necessary and sufficient for a so-called uniformly proper theory of continuous functions on the set Q, where the term, uniformly proper, is used in a sense likely to be admitted by anyone who gives the matter careful consideration.

26. If l is a prime of the form $4n + 3$, and p a prime of the form $ml + 1$, Jacobi and Cauchy have shown that a certain power of a prime ideal factor of p in the quadratic realm $k(\sqrt{-l})$ is always a principal ideal. Dirichlet later showed that the exponent is the number of classes of ideals in the realm. The result first mentioned has been extended by Eisenstein and Stickelberger to primes not of the form $ml + 1$. Professor Mitchell obtains a more general result by

considering a cyclotomic realm of degree $2m$, determined by $2m$ periods, each containing n lth roots of unity, where n is odd, and $l = 2mn + 1$. If p is any prime that in this realm is the product of $2m$ conjugate prime ideal factors, he shows that a certain power of the product of m of these factors, if properly chosen, is a principal ideal. The exponent in this case is identical with the number given by Kummer as the so-called first factor of the class number.

27. If l is a prime of the form $4n + 3$, Dirichlet has shown that the sum of the quadratic residues of l that lie between 0 and l is always less than the sum of the non-residues in the same interval. Certain limits between which these sums must lie have been given by Stern, and somewhat similar limits may readily be deduced from some results obtained by Holden. Professor Mitchell obtains much closer limits for these sums, by means of which he shows that their ratio approaches the limit 1, as l increases indefinitely. Similar results are obtained for higher residues and a generalization is given for composite moduli.

28. The usual theory of isothermal systems of curves is based on the Laplace equation $\varphi_{xx} + \varphi_{yy} = 0$, derived on the assumption that the heat is in equilibrium. Professor Kasner studies more general systems of curves (for example the curves of constant temperature in a weather map) where this assumption is not valid. In particular, the cases where the rate of change of the temperature with respect to the time is independent either of the time or of the position are investigated in detail. In the first case we are led to a certain type of linear doubly infinite systems of curves. In the second case we have, as in the case of equilibrium, merely a simply infinite system $\varphi(x, y) =$ constant, but the fundamental partial differential equation is $\varphi_{xx} + \varphi_{yy} = 1$.

29. Miss Cohen's paper is in abstract as follows: The conic determined by a line ξ and the four tangents at its intersections with a quartic curve is of degree five in the coefficients of the quartic and eight in ξ. By use of special cases it can be found in terms of known comitants of the quartic. The conic breaks up when ξ is such that three of the tangents are on a point; therefore the discriminant of the conic is the locus of such

lines. The singular lines of this locus are: (1) the twenty-four stationary lines of the quartic, which are double lines with one contact, the flex; (2) the twenty-eight double lines of the quartic, which are double lines with the same contacts as with the quartic; (3) the twenty-one lines for which four tangents meet on a point, which are quadruple lines. The first two of these account for all the common lines of the locus and the quartic.

30. Cesàro's theorem for an isogonal system of plane curves is to the effect that the osculating circles of the curves through a given point pass through a second point. A companion theorem for an equitangential system has been established by Scheffers (*Mathematische Annalen*, volume 60 (1905), page 491), namely, the osculating circles of the curves of such a system which touch a given line touch also a second line. In Professor Smith's paper it is shown that simple transformations of the surface elements satisfying a partial differential equation of the first order lead, in one case, to an isogonal system of surface bands, in the other to an equitangential system of these bands. For each point of a given surface band two circles exist, each in a plane normal to the band at the point, such that the first circle osculates all surfaces containing the band, and the second osculates the developable surface upon which the band lies. Consideration of the circles of the first class for an isogonal system of surface bands leads to a theorem analogous to Cesàro's, while for an equitangential system of bands the circles of the second class play the rôle of the osculating circles in Scheffer's theorem.

31. Professor Story gives a proof of the theorem: an infinite succession of rational or irrational real numbers $y_1, y_2, y_3, y_4, \cdots$ of which each is expressible in terms of the preceding three by an equation of the form

$$y_h = L_h y_{h-1} + M_h y_{h-2} + N_h y_{h-3} \quad \text{for } 4 \leqq h$$

converges to a definite finite limit if the first three y's are finite and the coefficients L_h, M_h, N_h, for each integral suffix h as great as 4 are real numbers that satisfy the conditions

$$L_h + M_h + N_h = 1, \quad 0 < N_h < L_h, \quad 0 \leqq M_h, \quad \alpha < L_h,$$

where α is an a priori given positive proper fraction that is

appreciably greater than 0. The convergence of a regular continued fraction of the second order (in the cases in which such a fraction is usable) follows from the special case of this theorem in which $\alpha = \frac{1}{3}$.

32. President Brown's address appeared in full in the February BULLETIN.

34. The most notable properties of the series of orthogonal or biorthogonal functions defined by linear differential equations of the second order and linear boundary conditions are: (1) the functions satisfy these boundary conditions, (2) they have a certain asymptotic form, and (3) the series may be used to represent wide classes of arbitrary functions.

The functions appearing in the series are also given as the solutions of a homogeneous linear integral equation of Fredholm type with the corresponding Green's function $G(x, y)$ for kernel.

Professor Birkhoff shows that, if $H(x, y)$ be any continuous kernel satisfying the given boundary conditions and possessing a discontinuous first derivative for $x = y$ like the Green's function, the solutions of the integral equation in H will form an orthogonal or biorthogonal series with properties (1), (2), (3), at least if simple further conditions be imposed. A converse statement is also established.

35. In the theory of linear difference equations two fundamental sets of solutions have been defined by means of their asymptotic properties in a half plane bounded by a straight line parallel to the axis of imaginaries and lying in the complex plane of the independent variables. Professor Birkhoff shows that similar sets of solutions exist for half planes bounded by any line not parallel to the axis of reals, and that these solutions may be made to take the place of the fundamental sets referred to.

36. Professor Miller's paper appears in full in the present number of the BULLETIN.

37. If p is a prime integer, $\alpha = e^{2i\pi/p}$, and $\mathfrak{p}$ is an ideal prime in the algebraic field Ω defined by α, then

$$\omega^{[N(\mathfrak{p})-1]/p} \equiv \alpha^{a} \pmod{\mathfrak{p}},$$

where a is some integer less than p, ω is an integer in Ω, and $N(\mathfrak{p})$ is the norm of $\mathfrak{p}$. Then we write

$$\alpha^a = \{\omega/\mathfrak{p}\}.$$

If Ω is a *regular* field and ω is a unit therein, then an explicit expression was given by Kummer for the exponent a. Let

$$\{\omega/\mathfrak{m}\} = \{\omega/\mathfrak{p}\}\{\omega/\mathfrak{q}\}\{\omega/\mathfrak{r}\}\cdots \quad (\mathfrak{m} = \mathfrak{p}\mathfrak{q}\mathfrak{r}\cdots),$$

where $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\cdots$, are prime ideals in any field Ω. Mr. Vandiver obtains in the present paper an explicit expression for $\{\omega/\mathfrak{m}\}$, where $\mathfrak{m}$ is a principal ideal in *any* cyclotomic field defined by a pth root of unity, and ω is included among certain types of units in the field.

38. Let $X_1, \cdots, X_r$ be the infinitesimal transformations of a finite continuous group with no exceptional infinitesimal transformation; and let $A_1, \cdots A_r$ be the matrices of any system whatever of independent infinitesimal linear homogeneous transformations generating a group with the same structure as G. The matrices $E_1, \cdots, E_r$ of the infinitesimal transformations of the group Γ adjoint to G constitute such a system. Denoting by SM the sum of the constituents in the principal diagonal of any matrix M, Professor Taber finds that $S(\Sigma\alpha_i A_i)^m$, for any positive integer m, is an invariant of the adjoint group Γ, and thus the coefficients of the characteristic equation of the general infinitesimal transformation $\Sigma\alpha_i A_i$ of the group $A_1, \cdots, A_r$ are invariants of Γ. Among these invariants one is of especial importance, namely, the quadratic form $S(\Sigma\alpha_i A_i)^2$. If the discriminant of this form is zero, and of nullity p, there are just p independent solutions of the equations $S(\Sigma\alpha_i A_i)A_k = 0$ $(k = 1, 2, \cdots, r)$; and if $\Sigma\alpha_i^{(q)} A_i$ for $q = 1, 2, \cdots, p$, are any p independent solutions of these equations, then $\Sigma\alpha_i^{(q)} X_i$, for $q = 1, 2, \cdots, p$, generate an invariant subgroup of G with p parameters. On the other hand, if the discriminant is not zero, the group G is perfect; and is either simple or semi-simple, unless possibly G contains an invariant subgroup every infinitesimal transformation of which is exceptional. For the case in which $A_1, \cdots, A_r$ are respectively identical with $E_1, \cdots, E_r$, these theorems are known. When the discriminant is not zero, the adjoint group is orthogonal, or becomes so by a suitable choice of the infinitesimal transformations of G; and in this case, if c_{ijk}

for $i, j, k = 1, 2, \cdots, r$, are the structural constants of G, we have $c_{ijk} + c_{ikj} = 0$.

F. N. COLE,
Secretary.

COMPLETE EXISTENTIAL THEORY OF THE POSTULATES FOR SERIAL ORDER.

BY PROFESSOR EDWARD V. HUNTINGTON.

(Read before the American Mathematical Society, December 27, 1916.)

THE purpose of this note is to establish the "complete independence"—in the sense defined by E. H. Moore*—of each of three different sets of postulates for serial order.

The first set of postulates (set A) is new and will be found more convenient for many purposes than either of the other sets. Set B dates back to Vailati, 1892.† Set C (a modification of set B and now widely used) was introduced by the present writer in 1905.‡

The universe of discourse considered in each of these sets is the universe of all systems (K, R), in which K is a class of elements, $A, B, C, \cdots$, and R is a dyadic relation; the notation $R(AB)$, or briefly AB, meaning that the relation R holds

* E. H. Moore, "Introduction to a form of general analysis," Yale University Press (1910), p. 82. An interesting example of a proof of complete independence is given by R. D. Beetle, "On the complete independence of Schimmack's postulates for the arithmetic mean," *Math. Annalen*, vol. 76 (1915), pp. 444–446. [Compare R. Schimmack, "Der Satz vom arithmetischen Mittel in axiomatischer Begründung," *Math. Annalen*, vol. 68 (1909), pp. 125–132.] For a similar discussion of an *almost* completely independent set of postulates, see L. L. Dines, "Complete existential theory of Sheffer's postulates for Boolean algebras," this BULLETIN, vol. 21 (1915), pp. 183–188. [Compare H. M. Sheffer, "A set of five independent postulates for Boolean algebras, with application to logical constants," *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 481–488.]

† G. Vailati, "Sui principî fondamentali della geometria della retta," *Rivista di Matematica*, vol. 2 (1892), pp. 71–75; B. Russell, Principles of Mathematics, vol. 1 (1903), pp. 203, 218–219.

‡ E. V. Huntington, "The continuum as a type of order," reprinted from the *Annals of Mathematics*, vols. 6 and 7 (1905), especially vol. 6, pp. 157–158; second edition, Harvard University Press, 1917, pp. 10–11, J. W. Young, Fundamental Concepts of Algebra and Geometry (1911), p. 68; A. N. Whitehead and B. Russell, Principia Mathematica, vol. 2 (1912), p. 513. (In the present terminology of Whitehead and Russell, a relation which satisfies postulate 1 is said to be "contained in diversity.").

for the elements A and B, in the order stated. The "postulates for serial order" are the conditions which such a system (K, R) must satisfy in order to be called a "serial system." If these conditions are satisfied, the relation R is called a serial relation with respect to the class K, and the notation $R(AB)$, or AB, may then be replaced by the more familiar notation $A \prec B$ ("A precedes B").

SET A. (POSTULATES 1, 2, 3, 4.)

The first postulate in this set concerns only a single element; the next two concern two distinct elements; and the fourth concerns three distinct elements.

Postulate 1. $AA \,.=.\, 0.$ (Irreflexiveness.)

That is, if A is an element of the class K, then the statement AA is always false.

Postulate 2. $A \neq B : \supset : AB \cdot \sim BA.$ (Connexity.)

That is, if A and B are distinct elements of K, then at least one of the statements AB and BA will be true.

Postulate 3. $A \neq B \,.\, AB \,.\, BA \,.=.\, 0.$

(Asymmetry for distinct elements.)

That is, if A and B are distinct elements of K, then not both the statements AB and BA can be true.

Postulate 4. $A \neq B \,.\, A \neq C \,.\, B \neq C \,.\, AB \,.\, BC : \supset : AC.$

(Transitivity for distinct elements.)

That is, if A, B, and C are distinct elements of K, then AB and BC together imply AC.

SET B. (POSTULATES 2, 3a, 4a.)

This set contains only three postulates, 3a and 4a being more inclusive forms of 3 and 4.

Postulate 2. $A \neq B : \supset : AB \cdot \sim BA.$ (Connexity.)

Postulate 3a. $AB \,.\, BA : = : 0.$

(Asymmetry for all elements.)

That is, if A and B are any elements of K (whether distinct or not), then not both the statements AB and BA can be true.

Postulate 4a. $AB \,.\, BC : \supset : AC.$

(Transitivity for all elements.)

That is, if A and B are any elements of K (whether distinct or not), then AB and BC together imply AC.

Here $3a$ implies 3 and 1, and $4a$ implies 4; also 1 and 3 imply $3a$, and 4 and 3 imply $4a$; so that sets A and B are clearly equivalent.

Note also that in set B, $4a$ may be replaced by 4, giving another set, B', comprising postulates 2, $3a$, and 4.

SET C. (POSTULATES 1, 2, $4a$.)

This set is a little simpler than set B, since postulate 1 is a little simpler than postulate $3a$; but neither set B nor set C is as explicit as set A.

Postulate 1. $AA \,.=.\, 0.$ (Irreflexiveness.)

Postulate 2. $A \neq B : \supset : AB \smile BA.$ (Connexity.)

Postulate $4a$. $AB \,.\, BC : \supset : AC.$

(Transitivity for all elements.)

Here 1 and $4a$ imply $3a$; and $3a$ implies 1; so that sets B and C are equivalent.

The question of the "complete independence" of each of these sets is in reality a question of classification. Every system (K, R), in the universe considered, either satisfies or fails to satisfy each of the postulates of each set. Now set A contains four postulates, and sets B and C each contain three. Hence *set A divides the universe theoretically into 16 compartments, while each of the sets B and C divides it into 8 compartments.* The purpose of this note is to show that no one of these compartments is "empty"; that is, to show that all the types of system (K, R) which can be distinguished by means of the postulates of any one of these three sets are actually represented among existing systems.

To prove this for set A, we exhibit 16 examples of systems (K, R); the first 8 of these answer also for sets B and C. In each of these examples, the class K is supposed to consist of only three elements, marked 1, 2, 3; the relation R is defined in each case by tabulating all the dyadic statements that are supposed to be true in that case. The character of each example is then shown in the table below, in which a dot (.) indicates that a postulate holds, and a cross (X) that it fails.

An examination of this table shows that examples of all the required types exist. (The entries below the double line are not necessary for the proof.)

In each of the examples shown, the class K consists of only

TABLE I.

Example.		Character.									
No.	Description of R. ($K = 1, 2, 3.$)	Set A.				Set B.			Set C.		
		1	2	3	4	2	3a	4a	1	2	4a
1	12, 13, 23.	.	.	.	.	.	.		.	.	
2	11, 22, 33, 12, 13, 23.	X	.	.	.	.	X	.	X	.	.
3	12.	.	X	.	.	X	.	.	.	X	.
4	11, 22, 33, 12.	X	X	.	.	X	X	.	X	X	.
5	12, 31, 23.	.	.	.	X	.	.	X	.	.	X
6	11, 22, 33, 12, 31, 23.	X	.	.	X	.	X	X	X	.	X
7	12, 23.	.	X	.	X	X	.	X	.	X	X
8	11, 22, 33, 12, 23.	X	X	.	X	X	X	X	X	X	X
9	12, 13, 23, 21, 31, 32.	.	.	X	.	.	X	X	.	.	X
10	11, 22, 33, 12, 13, 23, 21, 31, 32.	X	.	X	.	.	X	.	X	.	.
11	12, 21.	.	X	X	.	X	X	X	.	X	X
12	11, 22, 33, 12, 21.	X	X	X	.	X	X	.	X	X	.
13	12, 31, 23, 21.	.	.	X	X	.	X	X	.	.	X
14	11, 22, 33, 12, 31, 23, 21.	X	.	X	X	.	X	X	X	.	X
15	12, 13, 21, 31.	.	X	X	X	X	X	X	.	X	X
16	11, 22, 33, 12, 13, 21, 31.	X	X	X	X	X	X	X	X	X	X

three elements. It is easy to construct similar examples, however, in which K contains any number of elements, marked 1, 2, 3, $\cdots$ n, where n may be either finite ($n > 3$), or denumerably infinite. This may be done by adding the following items to the "descriptions of R" in the examples given above: in the examples which satisfy postulate 2, add xy whenever either x or y is > 3 and $x < y$; in examples 9 and 10 add also xy whenever either x or y is > 3 and $x > y$; in each of the even-numbered examples, add xx whenever $x > 3$. The character of these modified examples remains unchanged.

If we desire a *categorical* set of postulates for a *finite* series, we have merely to add to any one of our sets of postulates a postulate demanding that the system (K, R) shall contain exactly n elements ($n > 3$). The modified examples just

described show that each such categorical set of postulates will be also "completely independent."

HARVARD UNIVERSITY.

COMPLETE EXISTENTIAL THEORY OF THE POSTULATES FOR WELL ORDERED SETS.

BY PROFESSOR EDWARD V. HUNTINGTON.

(Read before the American Mathematical Society, December 27, 1916.)

A SYSTEM (K, R), where K is a class of elements $A, B, C, \cdots$ and R is a dyadic relation, is called a *well ordered* system when the following conditions are satisfied:*

(a) *the system* (K, R) *is a series; and*

(b) *every subsystem of* (K, R) *has a leading element.*†

Now when condition (b) is added to the conditions (a) which define a series, some of the conditions (a) become redundant. After eliminating these redundancies, we find the following *three sets of independent postulates for well ordered systems*, each of these three sets being in fact "completely independent" in the sense of E. H. Moore. (The numbering of the postulates is made to conform with that in the preceding note.)

SET I. (POSTULATES 1, 3, 5.)

Postulate 1. $AA \,.=.\, 0.$ (Irreflexiveness.)

Postulate 3. $A \neq B \,.\, AB \,.\, BA :=: 0.$ (Asymmetry for distinct elements.)

Postulate 5. *Every subsystem has at least one leading element.* ("Leadership," or the property of being "supplied with leaders.")

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* G. Cantor, *Math. Annalen*, vol. 49 (1897), p. 208. A. N. Whitehead and B. Russell, Principia Mathematica, vol. 3 (1913), p. 4.

† Here by a *series* we understand any system (K, R) which satisfies any one of the sets of postulates mentioned in the preceding note. A *subsystem* of (K, R) means any system (K', R') such that K' is a subclass of K, and $R' = R$. (Here K' is called a *subclass* of K if every element of K' belongs to K; that is, a subclass is either a part or the whole.) A *leading element* of a system means any element X having the following property: whenever Y is any other element of the system, then $R(XY)$, or simply XY, will be true. (If a system contains only a single element X, then X is a leading element of that system.)

SET II. (POSTULATES 3*a*, 5.)

Postulate 3*a*. $AB \,.\, BA : = : 0.$
(Asymmetry for all elements.)

Postulate 5. *Every subsystem has at least one leading element.*
(" Leadership.")

Here 3*a* implies 1 and 3; and 1 and 3 together imply 3*a*. Hence set II is equivalent to set I.

SET III. (POSTULATES 1, 5*a*.)

Postulate 1. $AA \,.\, = \,.\, 0.$ (Irreflexiveness.)

Postulate 5*a*. *Every subsystem has just one leading element.*
(" Unique leadership," or the property of being " supplied with unique leaders.")

Here 5*a* implies 3; for, if AB and BA were both true, and $A \neq B$, the subsystem consisting of the elements A and B would have two leading elements, contrary to 5*a*. Moreover, 3 and 5 imply 5*a*; for, if any subsystem had more than one leading element, say X_1 and X_2, then we should have X_1X_2 and X_2X_1, contrary to 3. Hence, set III is equivalent to set I.

It remains to show that every system that satisfies postulates 1, 3, and 5 will satisfy also the missing postulates for serial order, namely 2 and 4:

Postulate 2. $A \neq B : \supset : AB \cdot BA.$ (Connexity.)

Postulate 4. $A \neq B \,.\, A \neq C \,.\, B \neq C \,.\, AB \,.\, BC : \supset : AC.$
(Transitivity for distinct elements.)

Here 2 follows from 5. For, if neither AB nor BA were true, and $A \neq B$, then the subsystem composed of A and B would have no leading element.

Also, 4 follows from 3 and 5. For, if AC were false, then CA would be true, by 5; but from the truth of AB, BC, and CA, would follow the falsity of BA, CB, and AC, by 3, and hence the subsystem A, B, C would have no leading element.

Thus we see that any one of the sets I, II, III is equivalent to the usual requirement represented by (*a*) and (*b*).

Finally, each of the sets I, II, III is " completely independent " in the Moorean sense, as is shown by the following examples, selected from the list used for another purpose in the preceding note.

TABLE II.

Example (K, R).		Character.		
No.	Description of R. ($K=1, 2, 3.$)	Set I.	Set II.	Set III.
		1 3 5	3a 5	1 5a
1	12, 13, 23.	. . .	. .	. .
2	11, 22, 33, 12, 13, 23.	X . .	X .	X .
5	12, 31, 23.	. . X	. X	. X
6	11, 22, 33, 12, 31, 23.	X . X	X X	X X
9	12, 13, 23, 21, 31, 32.	. X .	X .	. X
10	11, 22, 33, 12, 13, 23, 21, 31, 32.	X X .	X .	X X
11	12, 21.	. X X	X X	. X
12	11, 22, 33, 12, 21.	X X X	X X	X X

An inspection of this table shows that all the types of systems required by Moore's "complete existential theory" for each of the three sets of postulates actually exist. (The entries below the double line in the table are not necessary for the proof.)

In conclusion we note that, by the same device as that used in the preceding note, each of these examples may be enlarged so as to contain n elements ($n > 3$, finite or denumerably infinite), without altering the character of the example. Hence we may readily obtain "categorical" sets of completely independent postulates for every *finite* well ordered system.

HARVARD UNIVERSITY.

GROUPS GENERATED BY TWO OPERATORS OF THE SAME PRIME ORDER SUCH THAT THE CONJUGATES OF ONE UNDER THE POWERS OF THE OTHER ARE COMMUTATIVE.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, December 28, 1916.)

§ 1. *Introduction.*

In a previous article,* devoted to a study of conjugate operators, attention was called to the fact that a complete set of conjugate operators which are not all relatively commutative is transformed under the group according to a substitution group which is at most *triply* transitive. It is easy to prove that this substitution group is at most *doubly* transitive, and, as the p conjugates of order 2 under the holomorph of the group of the prime order p are transformed under this holomorph according to a doubly transitive group, the substitution group in question can actually be doubly transitive.

The most general definition of a complete set of conjugate operators, or conjugate subgroups, of a group G is that the set is composed of the totality of operators, or subgroups, which correspond under one or more of the possible automorphisms of G. It is, however, usually assumed that such a set is composed of the totality of operators, or subgroups, which correspond under at least one of the *inner* automorphisms of G. Since every possible automorphism of any group is an inner automorphism under its holomorph, this difference in the possible definitions of the expression *complete set of conjugates* does not affect the general theorems relating to such sets.

The theorem mentioned in the first paragraph can be materially extended by the consideration of a complete set of conjugate operators involving an operator which is commutative with at least one of the others without being commutative with all of them. When this condition is satisfied the group of transformations of the set is always *simply* transitive. To prove this let s_1, s_2 represent two commutative operators of

* *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 19 (1910), p. 318.

the set of conjugates under consideration, while s_2, s_3 are two non-commutative operators of this set; then the group G containing these conjugates cannot involve any operator t which transforms s_1 into s_2 and also s_2 into s_3, since the equation

$$s_1 s_2 = s_2 s_1$$

implies

$$t^{-1}s_1 s_2 t = t^{-1}s_2 s_1 t, \quad \text{or} \quad t^{-1}s_1 t \cdot t^{-1}s_2 t = t^{-1}s_2 t \cdot t^{-1}s_1 t.$$

Hence the following theorem has been proved: *If a complete set of conjugate operators under any group G is transformed according to a multiply transitive substitution group by the operators of G then either every possible pair of these conjugates is composed of commutative operators or no one of these conjugates is commutative with any operator of the set besides itself.*

§ 2. *Conjugate Sets Having Special Properties.*

When one operator of a set of conjugate operators under a group G is commutative with each of the others except one, or with only one of the others, then it is clearly possible to arrange all the operators of the set in distinct pairs which constitute systems of imprimitivity of the substitution group according to which the operators of the set are transformed under G. In particular, it results from this that if a complete set of conjugate operators under a group is composed of an odd number of operators it is not possible that one operator be commutative with each of the others save one, or that one of these operators be commutative with one and only one other operator of the set.

If a complete set of conjugate operators under G contains two and only two operators which are not commutative with some one of these conjugates then these two operators must be commutative with each other unless the set of conjugates is transformed under G according to an imprimitive group having three elements in a system of imprimitivity. In particular, if the number of operators in a complete set of conjugates under G is not divisible by 3 and if this set involves two and only two operators which are not commutative with a certain other operator of the set, then these two operators must be commutative with each other. As a more special case we may note the following theorem: *If exactly $p - 2$, p being a*

prime number > 3, of the operators in a complete set of p conjugate operators are commutative with some one of these conjugates the remaining two operators of the set must be commutative with each other.

With slight modifications the arguments of the preceding paragraph apply to the case when in a complete set of conjugate operators under a group one of these conjugates is commutative with two and only two of the others. If these two operators are commutative with each other, the total set of conjugates is clearly transformed according to an imprimitive substitution group, having three elements in a system, by all the operators of the group in question. Hence it results that in a set of p conjugates under the powers of an operator of the prime order p it is not possible that one of these conjugates is commutative with two and only two of the others if these two are commutative and $p > 3$. In particular if a set of 5 conjugates under the powers of an operator of order 5 includes an operator which is commutative with at least one of the others without being commutative with each of them, then this operator is commutative with exactly two other operators and these two operators are non-commutative while the remaining two operators must be commutative.

If t transforms s into an operator which is commutative with s, then t^{-1} must evidently also transform s into an operator which is commutative with s. Hence *a complete set of conjugate operators under the powers of an operator of odd order must always involve an odd number of operators which are commutative with a particular operator of the set.* The preceding theorems relating to a complete set of conjugate operators clearly apply also to a complete set of conjugate subgroups if we replace the term commutative by the term permutable.

§ 3. *Determination of the Possible Groups Generated by Two Operators of the Same Prime Order such that the Conjugates of One under the Other are Commutative.*

Let s_1 and s_2 be two distinct operators of the same prime order p, and suppose that the p conjugates of s_2 under the powers of s_1 are commutative with each other. There are two possible cases to be considered. In the first case these p conjugates of s_2 are identical and the group generated by s_1 and s_2 is either the group of order p generated by s_2 or the

non-cyclic group of order p^2, as s_1 generates or does not generate s_2. This case is practically trivial and requires no further consideration.

In the second possible case the p conjugates of s_2 under the powers of s_1 are distinct. In this case the order of the group generated by s_1 and s_2 cannot be less than p^3 nor greater than p^{p+1}, since the p conjugates in question cannot generate a group whose order is less than p^2 or greater than p^p. When $p = 2$, the upper and lower limits for the orders of the possible groups are equal, and hence the octic group is the only possible group in this case. In general, *for every prime number p there are $p - 1$ groups which are generated by two non-commutative operators of order p satisfying the condition that the p conjugates of one of these operators under the powers of the other are commutative.*

Since the conjugates of s_2 under the powers of s_1 are commutative, it results that the powers of $s_2 s_1$ may be represented as follows, if $s_1 s_2 s_1^{-1} = t_1 s_2$, and $s_1 t_a s_1^{-1} = t_{a+1} t_a$:

$$s_2 s_1 = s_2 s_1, \quad (s_2 s_1)^2 = s_2 s_1 s_2 s_1^{-1} s_1^2 = t_1 s_2^2 s_1^2, \quad (s_2 s_1)^3 = t_2 t_1^3 s_2^3 s_1^3,$$

$$(s_2 s_1)^n = t_{n-1} t_{n-2}^n \cdots t_{n-r}^{\frac{n(n-1)\cdots(n-r+2)}{(r-1)!}} \cdots s_2^n s_1^n$$

$$(r = 2, 3, \cdots, n-1).$$

If we replace n by p in this formula it results directly that the order of $s_2 s_1$ is p whenever t_{n-1} is identity, that is, whenever the order of the commutative commutator subgroup of the group G generated by s_1, s_2 has less than $p - 1$ independent generators. Since the commutator subgroup of G is always of index p^2 it results that whenever the order of G is less than p^{p+1} it cannot contain any operator of order p^2, and that G always contains exactly $p + 1$ subgroups of index p.

When G is of maximal order, two of its subgroups of index p involve no operators of order p^2, but in the remaining $p - 2$ of these subgroups all the operators are of order p^2 except those found in their common subgroup, which is the commutator subgroup of G. Hence each of these groups of maximal order is generated by its operators of order p, and whenever $p > 2$ each of these groups is also generated by its operators of order p^2. In the special case when $p = 2$, these operators clearly generate only the cyclic subgroup of order 4.

The system of groups under consideration is of special interest because it includes $p-2$ non-abelian groups in which every operator except identity is of order p, for every possible value of the prime number p. The number of these groups for a particular prime number therefore depends upon this prime. The fact that the group generated by s_1, s_2 is completely determined by its order seems also worth noting.

UNIVERSITY OF ILLINOIS,
December, 1916.

A THEOREM IN THE ANALYSIS OF REAL VARIABLES.

BY PROFESSOR R. L. BORGER.

(Read before the American Mathematical Society, April 21, 1916.)

IN this paper the following theorem is proved:

THEOREM: *If the two real functions $U(x, y)$, $V(x, y)$ of the real variables x, y satisfy the following conditions at each point of a closed region R:*

(*a*) *U and V continuous in x and y jointly;*

(*b*) $$\frac{\partial U}{\partial x} \equiv U_1, \quad \frac{\partial U}{\partial y} \equiv U_2, \quad \frac{\partial V}{\partial x} \equiv V_1, \quad \frac{\partial V}{\partial y} \equiv V_2$$

exist and are finite;

(*c*) $\Delta U = hU_1 + kU_2 + \rho_1(h, k), \quad \Delta V = hV_1 + kV_2 + \rho_2(h, k);$

(*d*) $\operatorname{Lim}_{h,\,k=0} \dfrac{|\rho_i(h, k)|}{|h| + |k|} = 0 \quad (i = 1, 2);$

(*e*) $U_1 = V_2, \quad U_2 = -V_1;$

then U and V are analytic functions of x, y in R.

An immediate consequence of the theorem is that if any function W of a complex variable z possesses a finite derivative at each point of a simply connected closed region R then:

1. This derivative is continuous.
2. All the derivatives of W exist.
3. The function W may be represented by a power series in z.

To prove the theorem we apply the following, due to Kowalewski:*

If the two real functions $U(x, y)$, $V(x, y)$ are properly differentiable (satisfy hypotheses c, d) and if

$$U_1 = V_2$$

at each point of a rectangle R, there exists a function $\bar{U}(x, y)$ such that

$$\bar{U}_1 = U; \quad \bar{U}_2 = V.$$

The function U is, moreover, properly differentiable.

Employing this theorem we may infer the existence of two functions (in view of both parts of hypotheses (e)) $\bar{U}, \bar{V}$ such that

$$(1) \qquad \bar{U}_1 = U, \quad \bar{U}_2 = -V, \quad \bar{V}_1 = V, \quad \bar{V}_2 = U.$$

From equations (1)

$$(2) \qquad \bar{U}_1 = \bar{V}_2; \quad \bar{U}_2 = -\bar{V}_1.$$

Since $\bar{U}$ and $\bar{V}$ are properly differentiable we have by a second application of Kowalewski's theorem, two other functions, $\bar{\bar{U}}, \bar{\bar{V}}$, also properly differentiable and such that

$$(3) \qquad \bar{\bar{U}}_1 = \bar{U}; \quad \bar{\bar{U}}_2 = -\bar{V}; \quad \bar{\bar{V}}_1 = \bar{V}; \quad \bar{\bar{V}}_2 = \bar{U}.$$

The functions $\bar{U}(x, y)$, $\bar{V}(x, y)$, as well as their first and second partial derivatives, are easily seen to be continuous in the variables x, y and satisfy Laplace's differential equation $\Delta U = 0$, since

$$(4) \qquad \begin{aligned} \frac{\partial^2 \bar{U}}{\partial x^2} &= \frac{\partial \bar{U}_1}{\partial x} = \frac{\partial \bar{U}}{\partial x} = \bar{U}_1 = U, \\ \frac{\partial^2 \bar{U}}{\partial y^2} &= \frac{\partial \bar{U}_2}{\partial y} = -\frac{\partial \bar{V}}{\partial y} = -V_2 = -U. \end{aligned}$$

$$(5) \qquad \therefore \quad \frac{\partial^2 \bar{U}}{\partial x^2} + \frac{\partial^2 \bar{U}}{\partial y^2} = 0.$$

Similarly for $\bar{V}$.

The functions $\bar{U}, \bar{V}$ are then analytic† functions of x, y.

* Kowalewski, Die komplexen Veränderlichen und ihre Funktionen, p. 187.

† Picard, Traité d'Analyse, tome 2 (2d ed.), p. 18.

The functions U and V being contained among the derivatives of $\overline{U}$ and $\overline{V}$ are also analytic functions of x, y.

It may now be seen that we have the following representation for these functions:

$$U(x, y) = \sum_{k=0}^{\infty} (a_{10}x + a_{01}y)^k;$$

(6)

$$V(x, y) = \sum_{k=0}^{\infty} (b_{10}x + b_{01}y)^k;$$

$$((a_{10})^l(a_{01})^m = a_{lm}; \quad (b_{10})^l(b_{01})^m = b_{lm}; \quad l, m = 0, \cdots, k).$$

Differentiating the equations

$$U_1 = V_2, \quad U_2 = -V_1$$

$l - 1$ times with respect to x and $k - l$ times with respect to y, at the point (0, 0) we have the following relations among the coefficients:

$$(7) \quad a_{l,\,k-l} = b_{l-1,\,k-l+1}; \quad b_{l,\,k-l} = -a_{l-1,\,k-l+1} \quad (l = 1, \cdots, k).$$

We may now prove that if a function W of a complex variable z possesses a finite derivative at each point of a simply connected closed region, R,

1. This derivative is continuous.
2. All the derivatives of W exist.
3. The function W may be represented as a power series in z.

Since U and V are analytic functions of x and y, we see immediately from the representation

$$W = U(x, y) + iV(x, y)$$

that 1 and 2 are true. From equations (6)

$$W = U(x, y) + iV(x, y) = \sum_{k=0}^{\infty} (a_{10}x + a_{01}y)^k + i \sum_{k=0}^{\infty} (b_{10}x + b_{01}y)^k$$

$$= \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} [a_{l,\,k-l} + ib_{l,\,k-l}]x^l \cdot y^{k-l}.$$

From equations (7) we have

$$a_{l-1,\,k-l+1} + ib_{l-1,\,k-l+1} = i(a_{l,\,k-l} + ib_{l,\,k-l}) \quad (l = 1, \cdots, k)$$

and in general

$$a_{l-j,\,k-l+j} + ib_{l-j,\,k-l+j} = (i)^j(a_{l,\,k-l} + ib_{l,\,k-l}) \quad (j = 1, \cdots, l).$$

For $l = k$ it follows that

$$a_{k-j,\,j} + ib_{k-j,\,j} = (i)^j(a_{k0} + ib_{k0}) \quad (j = 1, \cdots, k).$$

Then

$$W(z) = \sum_{k=0}^{\infty} (a_{k0} + ib_{k0})(x + iy)^k$$

$$= \sum_{k=0}^{\infty} C_k z^k, \text{ where } (C_k \equiv a_{k0} + ib_{k0}).$$

This completes the theorem.

I am indebted to Professor E. J. Townsend for suggesting the problem.

OHIO UNIVERSITY,
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CONCERNING THE COMPLEMENT OF A COUNTABLE INFINITY OF POINT SETS OF A CERTAIN TYPE.

BY DR. J. R. KLINE.

(Read before the American Mathematical Society, December 27, 1916.)

IN his "Grundzüge der Mengenlehre," Hausdorff proved that if E denotes a euclidean space of two or more dimensions while R is a countable set of points belonging to E, then $E-R$ is a connected* point set.† It is the object of the present paper to prove a theorem, which contains Hausdorff's theorem as a special case. Hausdorff's method of proof does not apply for the proof of the more general theorem. While the proof is carried out for the case of two dimensions, it is evident that a similar proof would apply to any higher number of dimensions.

THEOREM. *If M is a domain‡ and G_1, G_2, G_3, $\cdots$ is a countable infinity of nowhere dense§ closed point sets, no one of*

* A set of points is said to be *connected* if, however it be divided into two mutually exclusive proper subsets, one of them contains a limit point of the other.

† Cf. F. Hausdorff, "Grundzüge der Mengenlehre," Leipzig, Veit, 1914, p. 333.

‡ A domain is a connected set of points M, such that if P is a point of M, then there exists a region containing P and lying in M.

§ A set of points is said to be *nowhere dense*, if in every region R_1 there exists a region R_2 entirely free of points of the set. A set is said to be *closed* if it contains all its limit points.

which disconnects any domain, then $M-(G_1+G_2+G_3+\cdots)$ *is connected.*

*Proof.**—Let A and B denote any two distinct points of $M-(G_1+G_2+\cdots)$.† The point set $M-G_1$ is a domain. For each point P of $M-G_1$, let K_{1P} denote a region, of subscript greater than or equal to 1, which belongs to the fundamental sequence‡ and is such that K'_{1P}§ is a subset of $M-G_1$. Call the set of all such regions S_1. There exists a simple chain‖ $R_{11}, R_{12}, R_{13}, \cdots, R_{1n_1}$ from A to B every link of which is a region of the set S_1. Call this chain C_1. As two regions which have a point in common, also have in common a region containing that point, and as G_2 is a nowhere dense closed point set, it follows that R_{1i} and R_{1i+1} $(i=1, 2, \cdots, n_1-1)$ have in common a point P_{1i}, not belonging to G_2. Call A, P_{10} and B, P_{1n_1}. For each point P of R_{1i} $(i=1,2,3,\cdots,n_1)$ which does not belong to G_2, let K_{2P} denote a region of the set $K_2, K_3, \cdots$, which is such that K'_{2P} lies in R_{1i} and contains no point of G_2. Call the set of all such regions S_2. It is now possible to construct a simple chain C_2, which satisfies all the requirements of Professor Moore's C_2¶ and has the additional property that every link of C_2 belongs to S_2 and therefore contains no points of G_1+G_2. Continue this process. We obtain an infinite sequence of chains $C_1, C_2, C_3, \cdots$ which satisfy all the requirements of Moore's sequence of chains $C_1, C_2, C_3, \cdots$ and have the additional property that no link of the chain C_n has a point in common with the set G_1+G_2*

* Our theorem is proved on the basis of the system of axioms Σ_1 proposed by R. L. Moore in his paper, "On the foundations of plane analysis situs," *Transactions Amer. Math. Soc.*, vol. 17 (1916), pp. 131–164.

† That $M-(G_1+G_2+G_3+\cdots)$ contains infinitely many points follows at once from one of the theorems of Baire, *Annali di Mat.* (3), vol. 3, p. 65.

‡ Select once for all a definite sequence, $K_1, K_2, \cdots$ satisfying the conditions of Axiom 1 of Σ_1. This definite sequence will be called the *fundamental sequence* and its regions will be termed *fundamental regions.*

§ The *boundary* of a point set M is the set of all limit points of M, which do not belong to M. If R is a region, R' denotes the point set composed of R plus its boundary.

‖ If A and B are distinct points, then a *simple chain* from A to B is defined by R. L. Moore as a finite sequence of regions $R_1, R_2, R_3, \cdots, R_n$ such that (1) R_i contains A if and only if $i=1$, (2) R_i contains B if and only if $i=n$, (3) if $1\leqq i\leqq n$ and $1\leqq j\leqq n$, while $i<j$, then R_i has a point in common with R_j if and only if $j=i+1$. The region R_k $(1\leqq k\leqq n)$ is said to be the *k*th *link* of the chain. See R. L. Moore, loc. cit., p. 134. For a proof of the existence of such a chain, see R. L. Moore, loc. cit., Theorem 10, p. 135.

¶ Cf. R. L. Moore, loc. cit., p. 137.

$+ \cdots + G_n$. Let $\overline{C}_n$ denote the point set which is the sum of all the links of the chain C_n, while C denotes the set of all points that the sets $\overline{C}_1$, $\overline{C}_2$, $\overline{C}_3$, $\cdots$ have in common. The point set C is a simple continuous arc* from A to B, lying entirely in the set $M - (G_1 + G_2 + \cdots)$.†

It follows that $M - (G_1 + G_2 + \cdots)$ is connected.

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AN ANALOGUE TO PASCAL'S THEOREM.

BY DR. A. L. MILLER.

A DECAGON is said to be doubly inscribed in a cubic if every odd side of the decagon cuts three even sides on the cubic and and every even side cuts three odd sides on the cubic.

That there exist decagons doubly inscribed in a cubic can be seen as follows. Let the decagon D have for sides $s_1, s_2, s_3, \cdots, s_{10}$ and let the cubic be C_3. Let

$$s_1 \text{ meet } s_{10}, s_2, s_4 \text{ on } C_3,$$

$$s_3 \text{ meet } s_2, s_4, s_6 \text{ on } C_3,$$

$$s_5 \text{ meet } s_4, s_6, s_8 \text{ on } C_3,$$

$$s_7 \text{ meet } s_6, s_8, s_{10} \text{ on } C_3,$$

while s_9 is the line joining the third intersection of s_8 with C_3 with the third intersection of s_{10} with C_3. Then, by Cayley's‡ theorem, s_9 also cuts s_2 on C_3.

By a repetition of this last theorem we obtain the following theorem analogous to Pascal's theorem:

If a decagon be doubly inscribed in a cubic the remaining ten intersections of the odd sides with the even ones lie on a conic.

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* If A and B are distinct points, a *simple continuous arc* from A to B is defined by Lennes as a bounded, closed, connected set of points containing A and B, but containing no proper connected subset containing both A and B. See N. J. Lennes, "Curves in non-metrical analysis situs with an application in the calculus of variations," *American Journal of Mathematics*, vol. 33 (1911) and this BULLETIN, vol. 12 (1906), p. 284.

† For a proof of this statement, see the proof of Theorem 15 of Moore's paper, loc. cit., pp. 136–9.

‡ Cayley: *Cambridge and Dublin Mathematical Journal*, vol. 3.

NOTES.

THE opening (January) number of volume 18 of the *Transactions of the American Mathematical Society* contains the following papers: "On the consistency and equivalence of certain definitions of summability," by W. A. HURWITZ and L. L. SILVERMAN; "The resolution into partial fractions of the reciprocal of an entire function of genus zero," by J. F. RITT; "On a general class of linear homogeneous differential equations of infinite order with constant coefficients," by J. F. RITT; "On the expressibility of a uniform function of several complex variables as the quotient of two functions of entire character," by T. H. GRONWALL; "Certain formal invariances in boolean algebras," by N. WIENER; "On a theory of linear differential equations in general analysis," by T. H. HILDEBRANDT; "Transformations T of conjugate systems of curves on a surface," by L. P. EISENHART.

Professor L. E. DICKSON has retired from the Editorial Committee, his unexpired term being filled by Professor L. P. EISENHART. Professors G. A. BLISS and E. B. WILSON have retired from the editorial staff and Professors C. N. MOORE and F. R. SHARPE have been appointed associate editors.

THE annual meeting of the London mathematical society was held November 2; Professor H. H. MACDONALD was elected president; the other officers were re-elected. The following papers were read at this meeting: by J. LARMOR, "Presidential address," and "Fourier harmonic analysis, its practical scope"; by W. H. YOUNG, "Multiple integration by parts and the second theorem of the mean"; by E. H. NEVILLE, "Moving axes and their uses in the differential geometry of euclidean space"; by J. H. HODGKINSON, "Areas and conformal representation."

AT the meeting held December 14 the following papers were presented: by W. BURNSIDE, "The efficiency of a surface of discontinuity regarded as a propeller"; by G. H. HARDY and S. RAMANUJAN, "Proof that almost all numbers are composed of about log log N prime factors," and "An asymptotic formula for the number of partitions of a number"; by G. N. WATSON, "The harmonic functions associated with a parabolic

cylinder (second paper)"; by D. BUCHANAN, "Orbits asymptotic to an isosceles triangle solution of the problem of three bodies"; by H. S. CARSLAW, "Diffraction of waves by a wedge of any angle"; by L. J. ROGERS, "Two theorems of combinatory analysis, and two allied identities"; by W. H. YOUNG and Mrs. YOUNG, "The internal structure of a set of points in space of any number of dimensions" and "The inherently crystalline structure of a function of any number of variables."

AT the meeting of the Edinburgh mathematical society on January 12 the following papers were read: by W. P. MILNE: "The apolar locus of two tetrads of points"; by L. R. FORD: "A method of solving algebraic equations."

AT the annual meeting of the British mathematical association held in London on January 5 the following papers were read: by T. P. NUNN, "The school syllabus in geometry"; by A. W. SIDDONS, "Some of the work of the teaching committee"; by A. N. WHITEHEAD, "Technical education and its relations to literature and science"; by A. W. SIDDONS, "An accuracy test set in some public schools"; by P. ABBOTT, "The place of mathematics in education reconstruction." Professor WHITEHEAD was elected president for the current year.

UNIVERSITY OF CHICAGO.—The following mathematical courses are announced for the summer quarter, June 18–August 31:—By Professor E. H. MOORE: The spectrum of limited infinite hermitian matrix (first half), four hours; Series (first half), four hours.—By Professor H. E. SLAUGHT: Differential equations, four hours.—By Professor E. J. WILCZYNSKI: Projective differential geometry, four hours.—By Professor J. W. A. YOUNG: Solid analytics, four hours.—By Professor A. C. LUNN: Relativity, four hours; Functions of a complex variable, four hours.—By Professor D. N. LEHMER (University of California): Synthetic projective geometry, four hours.—By Professor G. D. BIRKHOFF (Harvard University): Ordinary differential equations of the second order, four hours.

THE firm of G. E. Stechert and Company expect soon to have on sale a photographic reprint of the third edition of the

first volume of VALLÉE-POUSSIN's Cour d'Analyse. The book, in its original form, is extremely rare, very few copies having been sent out from Louvain before the destruction of the city.

PROFESSOR G. FUBINI, of the technical school at Turin, has been elected a corresponding member of the Accademia dei Lincei.

PROFESSOR E. LAURA, of the University of Messina, has been appointed associate professor of mechanics at the University of Padua.

DR. B. CALDONAZZO has been appointed docent in mechanics in the technical school at Milan.

MR. R. W. BARNARD has been appointed instructor in mathematics in the University of Michigan.

CHARLES J. WHITE, emeritus professor of mathematics in Harvard University, died February 12 at the age of 78 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOREL (E.). See VALLÉE-POUSSIN (C. DE LA).

CREMONA (L.). Opere matematiche. Tomo 3 con notizie della vita e delle opere dell'autore e con indice alfabetico per materie. Milano, Hoepli (Pisa, succ. fratelli Nistri), 1916. 4to. 22+520 pp. L. 30.00

FRAENKEL (A.). Ueber gewisse Teilbereiche und Erweiterungen von Ringen. Leipzig, Teubner, 1916. 8vo. 64 pp. M. 2.80

HANCOCK (H.). Elliptic integrals. (Mathematical monographs, No. 18.) New York, Wiley, 1916. 8vo. 125 pp. $1.25

JOSEPH (H. W. B.). An introduction to logic. 2d edition, revised and enlarged. New York, Oxford University Press, 1916. 8vo. 12+608 pp. $3.15

JOYCE (G. H.). Principles of logic. 2d edition. New York, Longmans, 1916. 8vo. 20+431 pp. $2.50

MANCINI (G.). L'opera "De corporibus regularibus" di Pietro Franceschi detto Della Francesca ursurpata da Fra Luca Pacioli. Memoria con tavole. (*Reale Accademia dei Lincei*, Anno CCCXII, 1915.) Roma, Pio Befani, 1915. 4to. 144 pp.+17 tavole.

PHILLIPS (H. B.). Differential calculus. New York, Wiley, 1916. 8vo. 5+162 pp. $1.25

PRINGSHEIM (A.). Vorlesungen über Zahlen- und Funktionenlehre. 1ter Band, 1te Abteilung: Reelle Zahlen und Zahlenfolgen. Leipzig, Teubner, 1916. 8vo. 12+292 pp. Geh. M. 12.00

SLAUGHT (H. E.). See WILCZYNSKI (E. J.).

STAGER (H. W.). A Sylow factor table of the first twelve thousand numbers, giving the possible number of Sylow sub-groups, of a group of given order between the limits of 0 and 12,000. Washington, Carnegie Institution, 1916. 12+120 pp.

VALLÉE-POUSSIN (C. DE LA). Intégrales de Lebesgue. Fonctions d'ensemble. Classes de Baire. Lecons professées au Collège de France. (Collection de monographies sur la théorie des fonctions publiée sous la direction de M. Emile Borel.) Paris, Gauthier-Villars, 1916. 8vo. 8+154 pp. Fr. 7.00

WILCZYNSKI (E. J.). College algebra with applications edited by H. E. Slaught. Boston, Allyn and Bacon, 1916. 8vo. 20+507 pp. Cloth. $2.00

II. ELEMENTARY MATHEMATICS.

ARNOLD (E. E.). See DURELL (F.).

BETZ (W.) and WEBB (H. E.). Plane and solid geometry. With the editorial cooperation of P. F. Smith. Boston, Ginn, 1916. 11+507 pp. Cloth. $1.36

BOOKMAN (C. M.). Business arithmetic. New York, American Book Company, 1914. 12mo. 250+12 pp. $0.65

COFFMAN (L. D.). See JESSUP (W. A.).

DEMING (A. G.). Number stories. Chicago, Beckley-Cardy Company, 1916. 205 pp. Cloth.

DURELL (F.) and ARNOLD (E. E.). Plane geometry. New York, Merrill, 1916. 300 pp. $0.88

HAMM (F. P.). Outline and suggestive methods and devices on the teaching of elementary arithmetic. Philadelphia, Lippincott, 1916. 5+40 pp. $1.00

JESSUP (W. A.) and COFFMAN (L. D.). Supervision of arithmetic. New York, Macmillan, 1916. 7+225 pp. Cloth.

MILNE (W. J.). Second course in algebra. New York, American Book Company, 1914. 12mo. 288 pp. Cloth. $0.88

——. Standard algebra, revised. New York, American Book Company, 1914. 12mo. 496+52 pp. $1.00

RIVENBURG (R. H.). A review of algebra. New York, American Book Company, 1914. 12mo. Cloth. 80 pp. $0.36

ROBBINS (E. R.). New plane and solid geometry. New York, American Book Company, 1916. 8vo. Cloth. 460 pp. $1.40. Solid geometry. 158 pp. $0.80. Plane geometry. 264 pp. $0.80

SMITH (P. F.). See BETZ (W.).

TAVOLE logaritmiche a cinque cifre decimali, raccolte e pubblicate per cura dell'Istituto idrografico della r. marina. 2a edizione. Genova, tip. Istituto idrografico della r. marina, 1916. 8vo. 27+524 pp.

THOMAS (A. O.). Rural arithmetic. New York, American Book Company, 1916. 8vo. 288 pp. Cloth. $0.68

WEBB (H. E.). See BETZ (W.).

WHEELER (A. H.). Examples in algebra. Boston, Little, Brown and Company, 1916. 14+257 pp. $0.90

III. APPLIED MATHEMATICS.

ARPESANI (C.). Elementi di tecnologia meccanica. Lavorazione dei metalli. 2a edizione rinuovata ed ampliata. Milano, Hoepli, 1915. 16+603 pp. L. 5.50

AUSTIN (F. E.). Examples in alternating currents. (For students and engineers.) Vol. 1. 2d edition. Hanover, N. H., F. E. Austin, 1916. 223 pp. Cloth. $2.00

BELDING (A. G.). Accounts and accounting practice. New York, American Book Company, 1916. 8vo. 224 pp. $0.90

BEXELL (J. A.) and NICHOLS (F. G.). Principles of book-keeping and farm accounts. New York, American Book Company, 1914. 8vo. 180 pp. $0.65

BRODIE (O. L.). See MORRISON (C. E.).

BROUGH (B. H.). A treatise on mine-surveying. 14th edition revised and enlarged by H. Dean. London, C. Griffin, 1916. 18+477 pp. 7s. 6d.

CAIN (W.). Earth pressure, retaining walls and bins. New York, Wiley, 1916. 297 pp. Cloth. $2.50

CARPENTER (C. K.). See WOLFARD (M. R.).

CLAUDI (C.). Manuale di prospettiva. 4a edizione riveduta. Milano, Hoepli, 1915. 12+76 pp. L. 2.50

CONTALDI (P.). La meccanica e le macchine nella scuola e nell'industria. 3a edizione riveduta ed ampliata. Milano, Hoepli, 1914–1916. 8vo. 16+676+16+768 pp. L. 12.00+12.00

DEAN (H.). See BROUGH (B. H.).

DEL FABRO (G.). Manuale di topografia per pratica e per studio. 3a edizione riveduta ed aumentata. Milano, Hoepli, 1914. 43+629 pp.

EFFEMERIDI astronomiche ad uso dei naviganti per l'anno 1917 (Istituto idrografico della r. marina). Genova, tip. Istituto idrografico della r. marina, 1916. 8vo. 187 pp. L. 1.50

FORBES (G.). David Gill: man and astronomer. Memories of Sir David Gill, astronomer (1879–1907) at Cape of Good Hope. Collected and arranged by G. Forbes. London, Murray, 1916. 11+418 pp. 12s.

GARUFFA (E.). L'aviazione. Milano, Hoepli, 1916. L. 8.50

HOLT (A. H.). Field astronomy. New York, Wiley, 1917. 8vo. 10+128 pp. Leather. $1.50

KAYE (G. W. C.) and LABY (T. H.). Tables of physical and chemical constants and some mathematical functions. 2d edition. New York, Longmans, 1916. 8vo. 8+153 pp. $2.00

KIDDER (F. E.) and NOLAN (T.). Architects' and builders' pocket book. 16th edition, rewritten. New York, Wiley, 1915. 1856 pp. $5.00

LABY (T. H.). See KAYE (G. W. C.).

LORENTZ (H. A.). The theory of electrons and its applications to the phenomena of light and radiant heat. 2d edition. Leipzig, Teubner, and New York, Stechert, 1916. 8vo. Geh. M. 9.00

MALAVASI (C.). Vademecum per l'ingegnere costruttore meccanico. 3a edizione notevolmente ampliata, con speciale riguardo alle esigenze dell'insegnamento tecnico. Milano, Hoelpi, 1916. 36+862 pp. L. 10.50

MARGUET (F.). Cours d'astronomie. Paris, Challamel, 1916. 8vo. Fr. 10.00

MIESSNER (B. F.). Radiodynamics. The wireless control of torpedoes and other mechanisms. New York, Van Nostrand, 1916. 206 pp. Cloth. $2.00

MORRISON (C. E.) and BRODIE (O. L.). Masonry dam design, including high masonry dams. 2d edition. New York, Wiley, 1916. 285 pp. Cloth. $2.50

NICHOLS (F. G.). See BEXELL (J. A.).

NOLAN (T.). See KIDDER (F. E.).

RAYMOND (W. G.). Plane surveying. New York, American Book Company, 1914. 12mo. 589 pp.+7 plates. Leather. $3.00

——. Railroad field manual for civil engineers. New York, Wiley, 1915. 7+398 pp. Leather. $3.00

RONCHETTI (G.). Grammatica del disegno. Metodo pratico per imparare il disegno. 2a edizione, interamente rifatta. Milano, Hoepli, 1915. 8vo. 8+225 pp.+un vol. di 80 tavole. L. 7.50

RORAY (N. L.). Industrial arithmetic. Philadelphia, Blakiston, 1916. 8+144 pp. $0.75

SPATARO (D.). Trattato teorico e pratico di idromeccanica. Volume 1. Milano, Hoepli, 1915. 8vo. 16+448 pp. L. 10.00

STEVENS (J. S.). Theory of measurements. A manual for physics students. New York, Van Nostrand, 1915. 8vo. 88 pp. $1.25

VIAPPIANI (A.). Trattato di idraulica pratica. Raccolta di formole e dati pratici da servire di guida nello studio nelle questioni relative al movimento delle acque, sia per utilizzarle in pro dell' agricoltura, industria, igiene e navigazione. . . . 2a edizione, riveduta e sensibilmente riformata. Milano, Hoepli, 1915. 32+740 pp.+14 tavole. L. 14.00

WOLFARD (M. R.) and CARPENTER (C. K.). The W-PVT chart. New York, Wiley, 1915. 24 by 38 inches. $0.50